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Pointwise Convergence and Uniformly convergence

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Abstract

In this paper, some basic definitions and theorems for convergence sequence are firstly presented. Secondly, a convergence sequence of numbers is described. Finally, the functional sequence of pointwise convergence and uniformly convergence are discussed.

1.1 Definition (Neighbourhood)

A neighbourhood of p is a set $N_r(p)$ consisting all q such that $d(p,q) < r$, for some r >0. The number r is called the radius of $N_r(p)$.

Note that in R^1, neighbourhood are segment and in R^2, neighbourhood are interior of circle.

1.2 Definition (Limit point of a set)

A point p is the limit point of the set E if every neighbourhood of p contain a point $q \in E$ with $p \neq q$.

1.3 Definition (Convergence of a sequence, limit)

A sequence $\{p_n\}$ in a metric space $X = (X, d)$ is said to converge or to be convergent if $\forall \varepsilon > 0, \exists N \ni n \geq N \Rightarrow d(p_n, p) < \varepsilon$.

We say that $\{p_n\}$ converge to p and p is the limit of sequence $\{p_n\}$.

Write $\quad p_n \to p \quad$ (or) $\quad \lim\limits_{n\to\infty} p_n = p \quad$ (or) $\quad \lim\limits_{n\to\infty} |p_n - p| = 0.$

1.4 Theorem

If p is a limit point of a set E, then every neighbourhood of p contain infinitely many points of E.

Proof:

The proof is by contraction.

Suppose there a nhd N of p which contain only a finite number of points of E.

Let $q_1, \ldots, q_n$ be those points of $N \cap E$ which are distinct from p.

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Put $r = \min_{1\le m\le n} d(p, p_m)$.

That is , r is the smallest of the numbers $d(p,q_1), \ldots , d(p,q_n)$.

The minimum of a finite set of positive numbers is clearly positive, so r > 0.

Now the nhd $N_r(p)$ contain no point q of E such that $p \neq q$. Thus we have shown that $\exists\ N_r(p) \ni N_r(p) \bigcap E = \phi$. So p is not a limit point of E.

This contradiction proves theorem.

1.5 Theorem

Let $\{p_n\}$ be a sequence in a metric space X with metric d.

(i) $\{p_n\}$ converges to p if and only if every neighbourhood of p contain p_n for all but finitely many n.

(ii) If $p \in X, p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.

(iii) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(iv)If $E \subseteq X$, and p is a limit point E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

Proof:

(i) The proof is similarly to (i) of Theorem 1.4.

(ii) Let $\varepsilon > 0$ be given.

Since $p_n \to p$ and $p_n \to p'$, $\exists N, N'$ such that

$$n \ge N \Rightarrow d(p_n, p) < \frac{\varepsilon}{2},$$

$$n \ge N' \Rightarrow d(p_n, p') < \frac{\varepsilon}{2}.$$

Hence if $n \ge \max(N, N')$, we have $d(p, p') \le d(p, p_n) + d(p_n, p') < \varepsilon$.

Since ε was arbitrary, $d(p, p') = 0$. Thus $p = p'$.

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(iii) Suppose $p_n \to p.$

Then for $\text{for}\,\varepsilon = 1, \exists N \ni n > N \Rightarrow d(p_n, p) < 1.$

Put $r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}.$

Then $d(x_n, x) \le r,\ \text{for}\, n = 1, 2, \ldots$ .Thus $\{p_n\}$ is bounded.

(iv)Since p is a limit point of E, for each positive integer n, there is a point $p_n \in E\,\text{such that}\, d(p_n, p) < \dfrac{1}{n}.$

Given $\varepsilon > 0,\text{choose}\, N\,\text{so that}\, N\varepsilon > 1.$

Thus if $n > N,\text{then}\, d(p_n, p) < \dfrac{1}{n} < \dfrac{1}{N} < \varepsilon.$

Hence $p_n \to p.$

1.6 Definition (Uniform Convergence)

A sequence of $\{f_n\}$ is said to be converge uniformly on E to a function f if $\forall \varepsilon > 0, \exists N = N(\varepsilon) \ni n \ge N \Rightarrow |f_n(x) - f(x)| < \varepsilon,\ \text{for all}\, x \in E.$

1.7 Definition (Pointwise Convergence)

A sequence of $\{f_n\}$ is said to be converge on E to a function f if $\forall \varepsilon > 0, \exists N = N(\varepsilon, x) \ni n \ge N \Rightarrow |f_n(x) - f(x)| < \varepsilon,\ \text{for all}\, x \in E.$

Note that the difference between these two concepts (Pointwise Convergence and Uniform Convergence) is the integer N depends on both ε and x for pointwise convergence and the integer N depends only on ε for uniform convergence.

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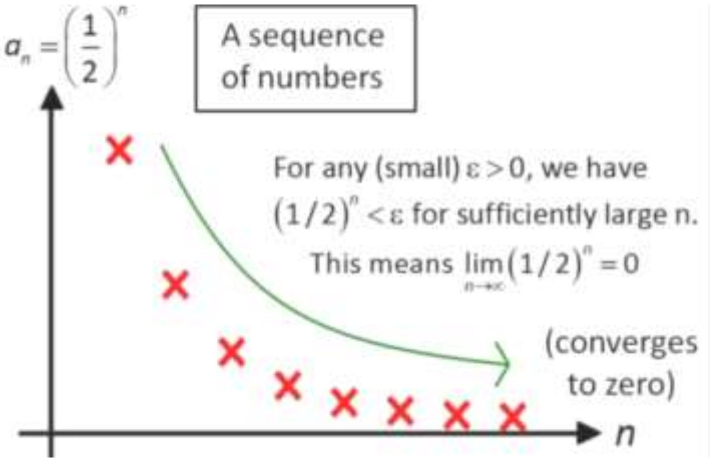
1.8 Example (A sequence of numbers)

$a_n = (\frac{1}{2})^n$

$a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{9}$,...



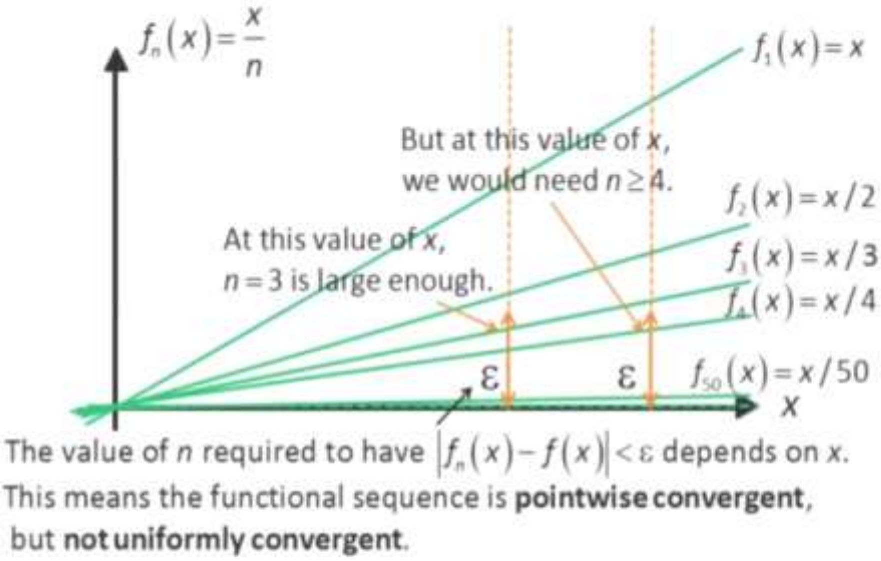
We have $\forall \varepsilon > 0, \exists N = N(\varepsilon) \ni n \geq N \Rightarrow \left|(\frac{1}{2})^n - 0\right| < \varepsilon.$

Thus $\lim_{n\to\infty} a_n = 0$.

1.9 Example (Functional sequence)

What is the functional sequence?

$\{f_n\}_{n\in N} = f_1 , f_2 , f_3 , \ldots$



These are functions, not numbers.

In general, $f_1(x) = x$, $f_2(x) = \frac{x}{2}$, $f_3(x) = \frac{x}{3}$, . . .

Suppose we have a fixed $x \in R$.

Then $\frac{x}{n} \to 0$ as $n \to \infty$.

We say that the functional sequence $\left\{\frac{x}{n}\right\}_n$ converges pointwise to $f(x) = 0$.

Suppose we have a fixed (small) value of ε.

How large does n need to be for us to have $|f_n(x) - f(x)| < \varepsilon$?

The answer to this question depends on x!

If a functional sequence is uniformly convergent, it must also be pointwise convergent with the same limit function.

But if it's pointwise convergent, it doesn't necessarily have to be uniformly convergent.

1.10 Example

Consider a new functional sequence defined on R:

$$\{f_n(x)\}_{n\in N} = \left\{x + \frac{1}{n}\right\}_{n\in N} = x+1 \,,\; x+\frac{1}{2} \,,\; x+\frac{1}{3} \,, \ldots$$

Questions:
Is this sequence pointwise convergent?
If so, what's the limit function?
And is it uniformly convergence?

Let us prove $\left\{x + \frac{1}{n}\right\}_n$ converge to $f(x) = x$.

(pointwise and uniformly) using the definition.

We have to show that a value of $N \in N$ can be chosen such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n > N$.

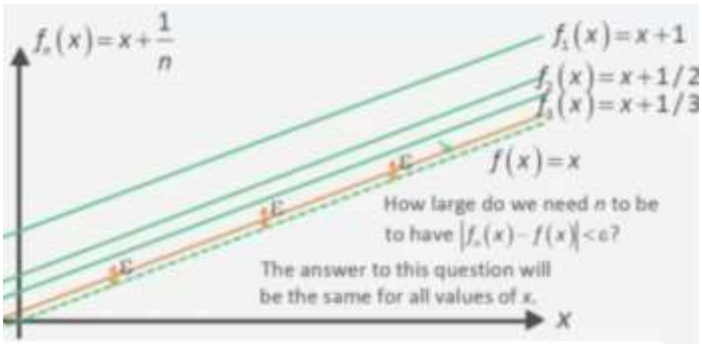
For pointwise convergence, N depends on ε and x. For uniform convergence, N can be depending on ε only.

We have $|f_n(x) - f(x)| = \left|x + \frac{1}{n} - x\right| = \frac{1}{n}$.

Choose N to be any natural number larger than $\frac{1}{\varepsilon}$. Then when n >N,

we have $|f_n(x) - f(x)| = \frac{1}{n} < \frac{1}{N} < \varepsilon$.

N depends only on ε, so this satisfies the definition of pointwise and uniform convergence simultaneously.



The sequence **converges pointwise** to $f(x) = x$ and it **converges uniformly** to $f(x) = x$ also.

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Reference

[1] Rudin, Walter; Functional Analysis, Mc Graw-Hill Series in Higher Mathematics, Mc Graw-Hill Book Co., New York, 1973.